P.G. (Sem-II) Maths
paper - VI
complex Analysis.

classmate
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Cauchy-Riemann equations:- A necessary condition that $w = f(z) = u(x,y) + iv(x,y)$ be analytic in a domain D is that u and v satisfy Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$ at every point of D

If these partial derivatives are also continuous, then Cauchy Riemann equations are sufficient conditions for f(z) to be analytic in D.

Theorem 1. Necessary condition for f(z) to be analytic.

$f(z) = u+iv$ is analytic in a domain D, then u, v satisfy the equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

provided the four partial derivatives u_x, u_y, v_x, v_y exist.

Proof:- Let $w = f(z) = u+iv$ be analytic in a domain D, then $\frac{dw}{dz}$ exists so that dw/dz has the same value along every path.

(i.) Along x-axis, $\delta z = \delta x$

$$\frac{dw}{dz} = \lim_{\delta z \to 0} \frac{\delta w}{\delta z} = \lim_{\delta x \to 0} \frac{\delta w}{\delta x} = \frac{\partial w}{\partial x}$$

(ii.) Along y-axis $\delta z = i\delta y$

$$\frac{dw}{dz} = \lim_{\delta z \to 0} \frac{\delta w}{\delta z} = \lim_{\delta y \to 0} \frac{\delta w}{i\delta y} = -i\frac{\partial w}{\partial y}$$

Equating ① & ②, $\frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}$

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

These equations are known as cauchy Riemann equations. Also we have seen that these partial derivatives exist.

A sufficient condition for $f(z)$ to be analytic.

Thm:- The function $w = f(z) = u + iv$ is analytic in a domain D if

(i.) u, v are differentiable in D and

$u_x = v_y, \; u_y = -v_x$

(ii.) the partial derivatives u_x, v_x, u_y, v_y all are continuous in D

Proof:- Let $w = f(z) = u + iv = u(x,y) + iv(x,y) = f(x,y)$ be s.t.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$u_x = v_y, \quad u_y = -v_x$$

Also let these derivatives be continuous.

Let the increments $\delta z, \delta u, \delta v, \delta w$ of z, u, v, w correspond to the increments $\delta x, \delta y$ of x and y

continuity of $u_x \Rightarrow \delta u = u_x \cdot \delta x + u_y \cdot \delta y + \alpha \delta x + \beta \delta y$

Similarly $\delta v = v_x \delta x + v_y \cdot \delta y + \alpha_1 \delta x + \beta_1 \delta y$

where $\alpha, \beta, \alpha_1, \beta_1$ all tend to zero as $\delta x \to 0, \delta y \to 0$ (3)

$$\frac{\delta w}{\delta z} = \frac{\delta u + i\delta v}{\delta x + i\delta y}$$

$$\delta u + i\delta v = \delta x(u_x + iv_x) + \delta y(u_y + iv_y) + (\alpha + i\alpha_1)\delta x + (\beta + i\beta_1)\delta y$$

$$= \delta x(u_x + iv_x) + i\delta y(-iu_y + v_y) + \alpha'\delta x + \beta'\delta y$$

where $\alpha' = \alpha + i\alpha_1, \beta' = \beta + i\beta_1$

using (1.) $\delta u + i\delta v = (u_x + iv_x)(\delta x + i\delta y) + \alpha'\delta x + \beta'\delta y$

Dividing by $\delta x + i\delta y$ and then using (2)

$$\frac{\delta w}{\delta z} = u_x + iv_x + \frac{\alpha'\delta x}{\delta x + i\delta y} + \frac{\beta'\delta y}{\delta x + i\delta y}$$

$$\left| \frac{\delta w}{\delta z} - (u_x + iv_x) \right| = \left| \frac{\alpha'\delta x}{\delta z} + \frac{\beta'\delta y}{\delta z} \right| \le |\alpha'| \cdot \left|\frac{\delta x}{\delta z}\right| + \beta' \left|\frac{\delta y}{\delta z}\right|$$

$$\le |\alpha'| + |\beta'| \text{ as } |\delta x| \le |\delta x + i\delta y|$$

or $\left| \frac{\delta w}{\delta z} - \frac{\partial w}{\partial x} \right| \le |\alpha| + |i\alpha_1| + |\beta| + |i\beta_1|$ as

$\alpha' = \alpha + i\alpha_1$

~~But when $\delta z \to 0$, the~~ R.H.S $\to 0$. from (3)

$$\lim_{\delta z \to 0} \frac{\delta w}{\delta z} - \frac{\partial w}{\partial x} = 0 \text{ or } \frac{dw}{dz} = \frac{\partial w}{\partial x} = u_x + iv_x$$

But u_x, v_x both exist. Hence dw/dz exists so that w is analytic in D.

Polar form of Cauchy's Riemann equations.

<u>Theorem:-</u> If $f(z) = u + iv$ is analytic function and $z = re^{i\theta}$ where u, v, r, θ are all real, Show that the Cauchy's Riemann equation are $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$, $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

or

Derive the necessary and sufficient conditions for $f(z)$ to be analytic in polar-coordinates.

<u>Proof:-</u> If $f(z) = u + iv$ is an analytic function then

$$u_x = v_y, \quad u_y = -v_x \quad - (1)$$

we have $x = r\cos\theta$, $y = r\sin\theta$

so, $x^2 + y^2 = r^2$ and $\tan\theta = y/x$

Diff. $x^2 + y^2 = r^2$ w.r.to x partially

$$2x + 0 = 2r\frac{\partial r}{\partial x}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta \quad (2)$$

Diff. w.r.to y partially

$$0 + 2y = 2r\frac{\partial r}{\partial y}$$

$$\text{or } \frac{\partial r}{\partial y} = \frac{y}{r} = \frac{\cancel{r}\sin\theta}{\cancel{r}} = \sin\theta \quad (3)$$

$\because \theta = \tan^{-1} y/x$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \frac{-y}{x^2}$$

$$= \frac{-y}{x^2+y^2} = -\frac{r\sin\theta}{r^2}$$

$$\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r} \quad \text{--- (4)}$$

and $$\frac{\partial \theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}} \times \frac{1}{x} = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2}$$

$$\frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r} \quad \text{--- (5)}$$

Again $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

or $$\frac{\partial u}{\partial x} = \cos\theta\cdot\frac{\partial u}{\partial r} + \frac{\partial u}{\partial \theta}\left(-\frac{\sin\theta}{r}\right)$$

or $$\frac{\partial u}{\partial x} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\cdot\frac{\partial u}{\partial \theta} \rightarrow (6)$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}$$

$$= \sin\theta\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\cdot\frac{\partial v}{\partial \theta} \quad (7)$$

$\because u_x = v_y$ or $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

from (6) and (7)

$$\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} = \sin\theta\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\cdot\frac{\partial v}{\partial \theta} \quad 7(A)$$

Further

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}$$

or $\dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial r}\sin\theta + \dfrac{\partial u}{\partial \theta}\cdot\dfrac{\cos\theta}{r}$ — (8)

$\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial v}{\partial \theta}\cdot\dfrac{\partial \theta}{\partial x}$

$= \dfrac{\partial v}{\partial r}\cos\theta + \dfrac{\partial v}{\partial \theta}\left(-\dfrac{\sin\theta}{r}\right)$

$= \dfrac{\partial v}{\partial r}\cos\theta - \dfrac{\sin\theta}{r}\dfrac{\partial v}{\partial \theta}$ — (9)

$\because \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ so from (8) & (9.)

$\sin\theta\dfrac{\partial u}{\partial r} + \dfrac{\cos\theta}{r}\cdot\dfrac{\partial u}{\partial \theta} = -\left\{\dfrac{\partial v}{\partial r}\cos\theta - \dfrac{\sin\theta}{r}\dfrac{\partial r}{\partial \theta}\right\}$

or $\sin\theta\dfrac{\partial u}{\partial r} + \dfrac{\partial u}{\partial \theta}\cdot\dfrac{\cos\theta}{r} = \dfrac{\sin\theta}{r}\dfrac{\partial v}{\partial \theta} - \dfrac{\partial v}{\partial r}\cos\theta$ (—10)

Multiplying (7A) by $\cos\theta$

$\cos^2\theta\dfrac{\partial u}{\partial r} - \dfrac{\sin\theta\cos\theta}{r}\dfrac{\partial u}{\partial r} = \sin\theta\cos\theta\dfrac{\partial v}{\partial r} + \dfrac{\cos^2\theta}{r}\cdot\dfrac{\partial r}{\partial \theta}$ (11)

Multiplying (10) by $\sin\theta$

$\sin^2\theta\dfrac{\partial u}{\partial r} + \dfrac{\partial u}{\partial \theta}\dfrac{\sin\theta\cos\theta}{r} = \dfrac{\sin^2\theta}{r}\dfrac{\partial v}{\partial \theta} - \dfrac{\partial v}{\partial r}\sin\theta\cos\theta$ (12)

Adding (11) and (12)

$\dfrac{\partial u}{\partial r}(\cos^2\theta + \sin^2\theta) = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}(\cos^2\theta + \sin^2\theta)$

or $\dfrac{\partial u}{\partial r}(1) = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$ proved

putting $\dfrac{\partial u}{\partial r}$ in (10) we have

$\sin\theta\cdot\dfrac{1}{r}\cancel{\dfrac{\partial v}{\partial \theta}} + \dfrac{\cos\theta}{r}\dfrac{\partial u}{\partial \theta} = -\cos\theta\dfrac{\partial v}{\partial r} + \cancel{\dfrac{\sin\theta}{r}\dfrac{\partial v}{\partial \theta}}$

or $\dfrac{\cancel{\cos\theta}}{r}\dfrac{\partial u}{\partial \theta} = -\cancel{\cos\theta}\cdot\dfrac{\partial v}{\partial r}$

or $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$ proved

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